MODALITIES, ABSTRACTION AND REASONING

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1. Introduction

One of the great advantages of natural languages is their ability to describe their own semantics and serve as their own metalanguage. This guarantees extensionality of the language, since new semantic rules can always be added. Natural languages achieve this ability with the aid of modal propositions, which define when simple propositions are true. Since we believe that it is very impractical to build an infinite hierarchy of languages (following Tarski), we are studying ways of using modalities to provide a formal language with the ability to define the truth of its own propositions (sacrificing, as natural languages do, enforced consistency of an arbitrary proposition, whereas consistency may be guaranteed on a level of useful logical subsystems; for example any logical system not containing self-referentially is free from paradoxes).

This work is part of an effort to develop a new logical formalism called natural logic, and to design and implement a computer language called Tecton, based upon natural logic. References [2,3,4,5,8] contain details of other aspects of natural logic and Tecton. Our plans for development include an interactive inference system based on natural logic.

A modality is an attribute; in the literature on philosophy and logic, a modality is often attributed with a proposition. A modality of a proposition is not necessarily related to the truth of the proposition; the truth itself is a modality of propositions. Introduction of modalities in logic gives an ability to express meta properties of the logic. It is thus a way to combine deduction and meta-deduction.

Just as modalities express truth values of propositions, it is also meaningful to talk about attributes of nouns, verbs etc. as modalities. For example, the propositions, "A proposition is true" and "A man is fat" have similar structure. The first one can be translated to "True(Proposition)" and the second one can be translated to "Fat(Man)."

A modality of propositions can also be viewed as a functor from propositions to propositions satisfying certain properties.

A proposition which describes a modality of some other proposition is called a modal proposition. Examples of modalities are: true, false, contrary, necessary, contingent, possible, impossible, provable, inconsistent, deterministic, absurd, meaningful, etc. A proposition may have more than one modality. For example, if "X" is true then "X" is possible. In representing knowledge for reasoning about systems (including real world systems), many other modalities, such as default (normal), probable, plausible, desirable, interesting, etc., turn out to be useful.

These notes discuss a general framework for introducing modalities into logic; modalities are classfied based on their properties and connections to each other. The difference between our approach and traditional approaches to modal logic (such as [1]) is that instead of concentrating on modal logic built around the pair of modalities (necessary, possible), we have tried to identify as many useful modalities as we can and build a natural classification of them. For example, we investigate the modality "normal," which captures a notion of default reasoning and is extremely useful in describing rules for dynamic hypothesis formation. The modal logic generated by this modality does not correspond to any standard modal theory (such as the S1 through S5 modal logics of C. S. Lewis, see [1]). (Unfortunately these notes do not yet contain a full discussion of the subject.)

2. What is a Modality?

As said above, a modality m is a functor from propositions to propositions. For any proposition p, m(p), or p is m, is the corresponding modal proposition using the modality m.

All modalities satisfy the following Axiom of Functionality:

For any modality m and proposition p, p is m or p is ~ m.

However, it should be noted that the Law of Excluded Middle, m(p) or m(~p), does not hold in general. The axiom of functionality is different from the axiom of excluded middle.

2.1 Derived Modalities

Given a modality m, it is possible to define three related modalities using just the connective ~ as follows.

negation : $N(m)(x) = \sim m(x)$

What is $N(N(m)) = \sim(N(m))\ (x) = \sim\sim m(x)$? Assumed to be $m(x)$.

opposite : $O(m)\ (x) = m(\sim x)$

similarly $O(O(m))\ (x) = O(m)\ (\sim x) = m(\sim\sim x)$? Assumed to be $m(x)$.

dual : $D(m)(x) = \sim m(\sim x)$, notice that $D(m) = N(O(m))$.

What is $D(D(m))$? $D(D(m))(x) = \sim D(m)\ (\sim x) = \sim(\sim m(\sim\sim x)) = ?\ m$?? Assumed to be $m(x)$.

Also $N(O(m))(x) = \sim O(m)(x) = \sim m(\sim x) = O(N(m))$?

<<<Does that mean that the negations on modalities and propositions are orthogonal and independent?>>>

The above three operators on modalities, negation, opposite, and duality, are idempotent.

As an example, let us define the above operations on the modality "necessary":

N(necessary) = unnecessary

O(necessary) = impossible

D(necessary) = possible.

We define the modality false as the opposite of modality true; note that false is not defined to be the negation of true.

The law of contradiction, ~(p and ~ p), in propositional calculus (which to capture the intuition that it is false that both p is true and false at the same time) translates to

false(true(p) and false(p))

Similarly the law of the excluded middle, p or ~p, in the propositional calculus (which says that it is true that either p or ~p is true) translates to

true(true(p) or true(~p)).

By the definition of true and false we have the following:

true(p) = false(~p) if we have ~~p = p. In that case,
false(p) = true(~p) = false(~~p).

3. Classification of Modalities

Regular modality: (i) for any proposition p, not both m(p) and m(~p),

(law of contradiction for modality)

(ii) |- p, then |- m(p),
(standard logic is stronger than properties of m)

(iii) m(p) and m(q) = m(p and q)
(monotonicity - not losing information)

A regular modality is strong but not the other way around.

Dual of regular: (i) law of excluded middle holds
(ii) |- m(p) then ~p is not derivable
(iii) or holds

Paradoxical Modality: m(p) |- m(~p) and m(~p) |- m(p).

Modalities described by programs: ...

Definition. A proposition x is m-meaningful if either m(x) or m(~x).

Definition. A modality m is weak if and only if all propositions are m-meaningful, so for any x, m(x) or m(~x).

The above is equivalent to requiring m to satisfy the following axiom

~m(~x) --> m(x)   (i)

So, possibly is a weak modality.

Two derived modalities can be introduced in case of a weak modality also as

contingency(x): m(x) and m(~x)

determinacy(x): ~m(x) or ~m(~x)

The dual of a weak modality can be defined in a similar way.

Definition: A modality m is called strong if and only if m(x) and m(~x) is always false.

The above is equivalent to requiring m satisfy the following axiom

$$m(x) \rightarrow \sim m(\sim x) \qquad \text{(ii)}$$

It should be obvious that the dual of a weak modality is a strong modality and vice versa. Because if m satisfies (ii), then d(m) satisfies (i) above as shown below:

$\sim d(m)(\sim x) \rightarrow d(m)(x)$, is equivalent to

$\sim\sim m(\sim\sim x) \rightarrow \sim m(\sim x)$, which is equivalent to

$m(x) \rightarrow \sim m(\sim x)$ assuming that $\sim\sim m$ is m and $\sim\sim x$ is x.

So, necessarily is a strong modality. Normally is also a strong modality.

For any strong modality, two derived modalities can be introduced as follows:

contingent(x) : $\sim m(\sim x)$ and $\sim m(x)$ - nondeterminism

determinacy(x): $m(\sim x)$ or $m(x)$ = $\sim$ contingency(x)

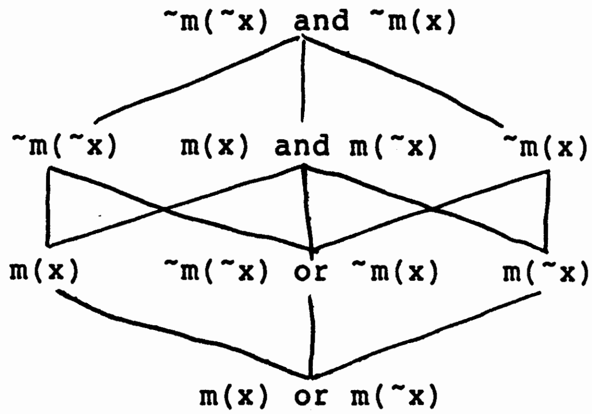
Can a modality be strong as well as weak? That would require that it satisfy both (i) and (ii), which amounts to

$$\sim m(\sim x) \leftrightarrow m(x)$$

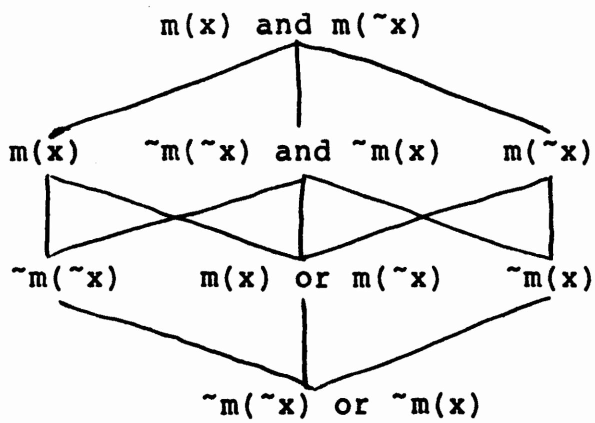
Modalities "true" and "false" satisfy this axiom as assuming usu~~s~~al interpretation of true and false. The above also means that d(m) and m are equivalent under the interpretation of ($\sim\sim m$ = m and $\sim\sim x = x$), meaning that the m is its own dual.

Is every modality either strong or weak? No; because contingency or determinacy are neither strong nor weak.

The lattice for weak modalities is:



The lattice for strong modalities is:



For each weak modality A there is a modality E, called A-contingent, such that "E that p" means "A that p, and A that not p". Normally, "contingent," without qualification, means possible-contingent.

For each strong modality A there is a modality F, called A-deterministic, such that "F that p" means "A that p, or A that not p".

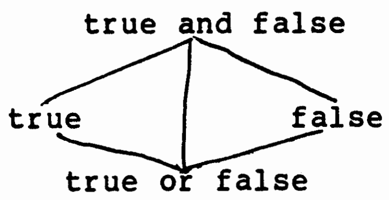
For each modality A there is meta-operation A-imply with the meaning that "X A-implies Y" means "it is A that X implies Y".

The following relations between particular modalities hold:

true= O(false)

false = O(true).

Lattice for true and false (implication going up)



A proposition is contrary if it is false for all instances.

contrary(x) = necessary(false(x)) ??

necessary(x) = derivable(x) and ~ derivable(~x)

necessary = derivable and dual(derivable)

contingent = ~ necessary and dual(necessary)

possible = dual(necessary)

deterministic(x) = necessary(x) or necessary(~x)

absurd(x) = (x implies ~ x and ~x implies x)

meaningful(x) = ~ absurd(x)

true implies possible

necessary implies true

normal(x) = (possible(x) implies true(x))

Axioms for possible and necessary: (taken from Hughes and Cres-well [1])

necessary(p) implies p

necessary(p implies q) implies (necessary(p) implies necessary(q))

Rule of inference:

from a theorem p, deduce necessary(p)

We can derive the following rule of inference

from a theorem p implies q, deduce
necessary(p) implies necessary(q),
and pos(p) implies pos(q).

Normal is a regular modality satisfying the axiom:

normal(p) eqv (pos(p) implies p).

Properties of norm using and and or:

normal(p or q) eqv (pos(p or q) implies (p or q))

eqv (pos(p) or pos(q)) implies (p or q)

implies (pos(p) implies p) or (pos(q) implies q)

implies normal(p) or normal(q)

(opening up, get a propositional tautology)

normal(p and q) eqv (pos(p and q) implies (p and q))

implies (pos(p) and pos(q)) implies (p and q)

does it imply normal(p) and normal(p)??? - no.

normal(p) and normal(q) implies normal(p and q).

4. Refinement and Modalities

For all non-modal propositions there is one general law of abstraction: if a proposition is true for a subject X, then it is true for any concretization of X. However it is not necessarily true for modal propositions. For example, the proposition "it is possible that a scientist becomes a manager" does not entail the proposition "it is possible that I become a manager". The following table shows how a modality of proposition may change from an abstraction of X to X (it shows only modalities which are in the "necessary" lattice):

An abstraction of X	X
meaningful	meaningful
meaningful	possible
meaningful	unnecessary
meaningful	contingent
meaningful	deterministic
meaningful	necessary
meaningful	impossible
meaningful	absurd
possible	possible
possible	necessary
possible	impossible
possible	contingent
possible	absurd
possible	deterministic
unnecessary	necessary
unnecessary	impossible
unnecessary	unnecessary
unnecessary	contingent
unnecessary	deterministic
unnecessary	absurd
contingent	contingent
contingent	deterministic
contingent	necessary
contingent	impossible
contingent	absurd
deterministic	deterministic
deterministic	necessary
deterministic	impossible
deterministic	absurd
necessary	necessary
necessary	absurd
impossible	impossible
impossible	absurd
absurd	absurd

5. Real and nominal modal propositions

There are two types of modal propositions: real and nominal. A real proposition has the same form as a categorical proposition, except that it has a modal adverb associated with the copula. The meaning of a real modal proposition is that all given instances of the subject are predicated with given predicate with given modality.

The form of a nominal modal proposition is "It is M that P," where M is a modality and P is a proposition, or simply "M, P" where M is modal adverb. The meaning of a nominal modal proposition is that the subject is predicated with given predicate with given modality.

For instance, "people in this room normally don't smoke" is a real normal proposition, while "normally, people in this room don't smoke" is a nominal normal proposition.

If a nominal modal proposition implies a real modal proposition with the same modality, subject and predicate, then the converse is true for the dual modality. If a nominal necessary proposition is true, then a real necessary proposition with the same subject and predicate is true also.

6. Things to include and open issues

Further work along these lines will include:

1. General classification of modalities.
2. Axiomatic theories for most important modalities.
3. Free models for logical systems containing modalities.
4. Methods of automated inference for modalities.

7. References

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